

Two-parameter asymptotics in the Cauchy problem for a parabolic equation

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Abstract. The Cauchy problem for a quasi-linear parabolic equation with a small parameter at a higher derivative is considered. The initial step-like function contains another small parameter. Formal asymptotic solutions of the problem in small parameters are constructed.

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1 Introduction

In the present work, we consider the Cauchy problem for a quasi-linear parabolic equation:

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad t \geq 0, \quad \varepsilon > 0, \quad (1.1)$$

$$u(x, 0, \varepsilon, \rho) = \nu(x\rho^{-1}), \quad x \in \mathbb{R}, \quad \rho > 0. \quad (1.2)$$

We assume that $\varepsilon > 0$, the function φ is infinitely differentiable and its second derivative is strictly positive. The initial function ν is bounded and smooth.

This model is used for studying the evolution of a wide class of physical systems with a small dissipation and probabilistic processes [1, 2, 3]. The interest to problem under consideration is explained by physical applications and the fact that its solutions allow one to obtain viscous generalized solutions of the limit equation. This problem had been studied by N.S. Bakhvalov, I.M. Gelfand, A.M. Il'in, E. Hopf, O.A. Ladyzhenskaya, O.A. Oleinik, and many other mathematicians. It is strictly proved [4], that there exists a unique bounded infinitely differentiable with respect to x and t solution $u(x, t, \varepsilon)$.

The aim of the present paper is to construct asymptotics solutions $u(x, t, \varepsilon, \rho)$ of problem (1.1)–(1.2) as $\varepsilon \rightarrow 0$ and $\rho \rightarrow 0$. The structure of asymptotic series essentially depends on the relation between parameters ε and ρ as shown below.

First, note that the change of variables

$$x = \rho\sigma, \quad t = \rho\theta$$

in equation (1.1) and the initial condition (1.2) leads to the following problem:

$$\frac{\partial u}{\partial \theta} + \frac{\partial \varphi(u)}{\partial \sigma} = \delta \frac{\partial^2 u}{\partial \sigma^2}, \quad \delta = \frac{\varepsilon}{\rho}, \quad u(\sigma, 0) = \nu(\sigma).$$

An asymptotic approximation of the solution of such a problem up to an arbitrary power of parameter δ is obtained directly from [3, Ch. VI]. For example, the expansion of the solution

in a neighborhood of the singular point $(x, t) = (0, \rho)$ has the form

$$\sum_{k=1}^{\infty} \delta^{k/4} \sum_{j=0}^{k-1} w_{k,j}(\xi, \tau) \ln^j \delta,$$

where coefficients $w_{k,j}(\xi, \tau)$ depend on the inner variables, which are determined using change

$$x = \varepsilon^{3/4} \rho^{1/4} \xi, \quad t = \rho + \varepsilon^{1/2} \rho^{1/2} \tau.$$

The leading term of this expansion is found with the help of the Cole–Hopf transform:

$$w_{1,0}(\xi, \tau) = -\frac{2}{\varphi''(0)\Lambda(\xi, \tau)} \frac{\partial \Lambda(\xi, \tau)}{\partial \xi},$$

$$\Lambda(\xi, \tau) = \int_{-\infty}^{\infty} \exp(-2z^4 + z^2\tau + z\xi) dz.$$

Now, let the relation between parameters ε and ρ be such that

$$\frac{\rho}{\varepsilon} \rightarrow 0.$$

Assume that the function ν in (1.2) satisfies the following asymptotic relations:

$$\nu(\sigma) = \sum_{n=0}^{\infty} \frac{\nu_n^{\pm}}{\sigma^n}, \quad \sigma \rightarrow \pm\infty. \quad (1.3)$$

In papers [5, 6], it is shown that for the solution of problem (1.1)–(1.2) as $\varepsilon \rightarrow 0$, $\rho \rightarrow 0$, $\mu = \rho/\varepsilon \rightarrow 0$ in the strip

$$\{(x, t) : x \in \mathbb{R}, 0 \leq t \leq T\}$$

there holds the asymptotic formula

$$u(x, t, \varepsilon, \rho) = h_0\left(\frac{x}{\rho}, \frac{\varepsilon t}{\rho^2}\right) - R_{0,0,0}\left(\frac{x}{2\sqrt{\varepsilon t}}\right) + \Gamma\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) + O(\mu^{1/2} \ln \mu),$$

where

$$h_0(\sigma, \omega) = \frac{1}{2\sqrt{\pi\omega}} \int_{-\infty}^{+\infty} \nu(s) \exp\left[-\frac{(\sigma-s)^2}{4\omega}\right] ds,$$

$$R_{0,0,0}(z) = \frac{\nu_0^-}{\sqrt{\pi}} \int_z^{+\infty} \exp(-y^2) dy + \frac{\nu_0^+}{\sqrt{\pi}} \int_{-\infty}^z \exp(-y^2) dy,$$

Γ is the solution of the equation

$$\frac{\partial \Gamma}{\partial \theta} + \frac{\partial \varphi(\Gamma)}{\partial \eta} - \frac{\partial^2 \Gamma}{\partial \eta^2} = 0$$

in the inner variables $\eta = x/\varepsilon$, $\theta = t/\varepsilon$ with the initial condition

$$\Gamma(\eta, 0) = \begin{cases} \nu_0^-, & \eta < 0, \\ \nu_0^+, & \eta \geq 0. \end{cases}$$

In the present paper, formal asymptotic solutions of problem (1.1)–(1.2) are constructed in the form of infinite series.

2 The outer expansion

The behavior of the solution of problem (1.1)–(1.2) is mainly determined by the solution of the limit problem

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = 0, \quad u(x, 0) = \begin{cases} \nu_0^-, & x < 0, \\ \nu_0^+, & x \geq 0. \end{cases} \quad (2.1)$$

For $\nu_0^- > \nu_0^+$ using the method of characteristics, we find its generalized solution

$$u_{0,0}(x, t) = \begin{cases} \nu_0^-, & x < ct, \\ \nu_0^+, & x > ct, \end{cases} \quad c = \frac{\varphi(\nu_0^+) - \varphi(\nu_0^-)}{\nu_0^+ - \nu_0^-}.$$

This solution is discontinuous on the line of the shock wave $x = ct$.

First, let us find the outer expansion in the domain

$$\Omega_0^+ = \{(x, t) : x > ct + \varepsilon^{1-\delta_0}, 0 < \delta_0 < 1\}.$$

Taking into account (1.3), we will construct the outer asymptotic expansion in the form of the series

$$U^+(x, t, \varepsilon, \rho) = \nu_0^+ + \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \rho^{m-n} \varepsilon^n u_{m,n}^+(x, t). \quad (2.2)$$

In the domain

$$\Omega_0^- = \{(x, t) : x < ct - \varepsilon^{1-\delta_0}\}$$

we will construct an analogous series

$$U^-(x, t, \varepsilon, \rho) = \nu_0^- + \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} \rho^{m-n} \varepsilon^n u_{m,n}^-(x, t). \quad (2.3)$$

Formally substituting series (2.2) and (2.3) into equation (1.1) and collecting coefficients at $\rho^{m-n} \varepsilon^n$, we arrive at the recurrence system of initial value problems

$$\frac{\partial u_{m,n}^{\pm}}{\partial t} + \varphi'(\nu_0^{\pm}) \frac{\partial u_{m,n}^{\pm}}{\partial x} = F_{m,n}^{\pm}, \quad u_{m,n}^{\pm}(x, 0) = \delta_{n,0} \nu_m^{\pm} x^{-m}, \quad (2.4)$$

where $\delta_{0,0} = 1$, $\delta_{n,0} = 0$ for $n \neq 0$,

$$F_{m,n}^{\pm} = \frac{\partial^2 u_{m-1,n-1}^{\pm}}{\partial x^2} - \sum_{q=2}^{m-n} \frac{\varphi^{(q)}(\nu_0^{\pm})}{q!} \sum_{\substack{i_1+\dots+i_q=m \\ j_1+\dots+j_q=n}} \frac{\partial (u_{i_1,j_1} \cdot \dots \cdot u_{i_q,j_q})}{\partial x}. \quad (2.5)$$

Using the method of characteristics, we find the coefficients of the outer expansion:

$$u_{m,n}^{\pm}(x, t) = \frac{\delta_{n,0} \nu_m^{\pm}}{[x - \varphi'(\nu_0^{\pm})t]^m} + \int_0^t F_{m,n}^{\pm}(x - \varphi'(\nu_0^{\pm})(t - t'), t') dt'. \quad (2.6)$$

Thus, for $t = 0$ formal series (2.2) and (2.3) become asymptotic series for the initial function:

$$U^{\pm}(x, 0, \varepsilon, \rho) = \sum_{m=0}^{\infty} \nu_m^{\pm} \left(\frac{\rho}{x}\right)^m.$$

Using relations (2.5) and (2.6), by induction we arrive at the following statement.

Theorem 1. *For $m \geq 1$ and $0 \leq n \leq m-1$, there holds formula*

$$u_{m,n}^{\pm}(x, t) = \sum_{s=n}^{m-1} \frac{\alpha_{m,n,s}^{\pm} t^s}{[x - \varphi'(\nu_0^{\pm})t]^{m+s}}, \quad (2.7)$$

where $\alpha_{m,n,s}^{\pm}$ are constants.

3 The inner expansion

We make the change of variables

$$x = \rho\sigma, \quad t = \frac{\rho^2}{\varepsilon}\omega. \quad (3.1)$$

Then equation (1.1) becomes

$$\frac{\partial h}{\partial \omega} - \frac{\partial^2 h}{\partial \sigma^2} = -\mu \frac{\partial \varphi(h)}{\partial \sigma}, \quad (3.2)$$

where $h(\sigma, \omega) \equiv u(\rho\sigma, \rho^2\omega/\varepsilon)$,

$$\mu = \frac{\rho}{\varepsilon} \rightarrow 0.$$

We seek the inner expansion in the form of the series

$$H(\sigma, \omega, \mu) = \sum_{n=0}^{\infty} \mu^n h_n(\sigma, \omega), \quad (3.3)$$

for whose coefficients from equation (3.2) and condition (1.2) we obtain the recurrence chain of initial value problems

$$\frac{\partial h_0}{\partial \omega} - \frac{\partial^2 h_0}{\partial \sigma^2} = 0, \quad h_0(\sigma, 0) = \nu(\sigma), \quad (3.4)$$

$$\frac{\partial h_1}{\partial \omega} - \frac{\partial^2 h_1}{\partial \sigma^2} = -\frac{\partial \varphi(h_0)}{\partial \sigma}, \quad h_1(\sigma, 0) = 0, \quad (3.5)$$

$$\frac{\partial h_n}{\partial \omega} - \frac{\partial^2 h_n}{\partial \sigma^2} = -\frac{\partial E_n}{\partial \sigma}, \quad h_n(\sigma, 0) = 0, \quad (3.6)$$

where

$$E_n = \sum_{q=1}^{n-1} \frac{\varphi^{(q)}(h_0)}{q!} \sum_{n_1+\dots+n_q=n-1} \prod_{p=1}^q h_{n_p}, \quad n \geq 2.$$

To find where series (3.3) makes sense and to construct asymptotics in other domains, it is necessary to know the behavior of functions $h_n(\sigma, \omega)$ at infinity. As shown in paper [7], the function

$$h_0(\sigma, \omega) = \frac{1}{2\sqrt{\pi\omega}} \int_{-\infty}^{\infty} \nu(s) \exp \left[-\frac{(\sigma-s)^2}{4\omega} \right] ds, \quad (3.7)$$

i.e., the solution of problem (3.4), has the following asymptotics as $|\sigma| + \omega \rightarrow \infty$:

$$h_0(\sigma, \omega) = R_{0,0,0}(z) + \sum_{m=1}^{\infty} \omega^{-m/2} [R_{0,m,0}(z) + \ln \omega R_{0,m,1}(z)], \quad (3.8)$$

where

$$R_{0,0,0}(z) = \nu_0^- \operatorname{erfc}(z) + \nu_0^+ \operatorname{erfc}(-z), \quad (3.9)$$

$$\operatorname{erfc}(z) = \frac{1}{\sqrt{\pi}} \int_z^{+\infty} \exp(-y^2) dy, \quad z = \frac{\sigma}{2\sqrt{\omega}},$$

$R_{0,m,0}$ and $R_{0,m,1}$ are smooth functions.

The asymptotics of solutions to problems (3.5)–(3.6), which can be expressed in the form of convolution

$$h_n(\sigma, \omega) = - \int_0^\omega \int_{-\infty}^\infty \frac{1}{2\sqrt{\pi(\omega-v)}} \exp\left[-\frac{(\sigma-s)^2}{4(\omega-v)}\right] \frac{\partial E_n}{\partial s} ds dv, \quad (3.10)$$

as $|\sigma| + \omega \rightarrow \infty$ are found by the same method of paper [7]. Proceeding by induction, one can show that the following statement is valid.

Theorem 2. *For solutions of problems (3.4)–(3.6), which are determined recursively by formulas (3.7) and (3.10), for all $n \geq 0$ there holds the asymptotic expansion*

$$h_n(\sigma, \omega) = \omega^{n/2} \sum_{m=0}^{\infty} \omega^{-m/2} \sum_{l=0}^m (\ln \omega)^l R_{n,m,l}(z). \quad (3.11)$$

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